Figure 3.37 The graph of $f(x) = \dfrac{x^4}{x^2 + 1}$

7 Identify slant asymptotes.
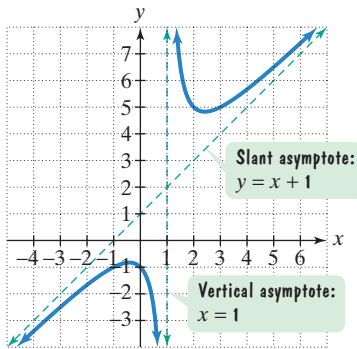
x		-2	-1	1	2
$f(x) = \dfrac{x^4}{x^2 + 1}$		$\dfrac{16}{5}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{16}{5}$

Step 7 Graph the function. Figure 3.37 shows the graph of f using the points obtained from the table and y-axis symmetry. Notice that as x approaches infinity or negative infinity ($x \to \infty$ or $x \to -\infty$), the function values, $f(x)$, are getting larger without bound $[f(x) \to \infty]$. ●

✓ **Check Point 7** Graph: $f(x) = \dfrac{x^4}{x^2 + 2}$.

Slant Asymptotes

Examine the graph of

$$f(x) = \frac{x^2 + 1}{x - 1},$$

shown in **Figure 3.38**. Note that the degree of the numerator, 2, is greater than the degree of the denominator, 1. Thus, the graph of this function has no horizontal asymptote. However, the graph has a **slant asymptote,** $y = x + 1$.

  The graph of a rational function has a slant asymptote if the degree of the numerator is one more than the degree of the denominator. The equation of the slant asymptote can be found by division. For example, to find the slant asymptote for the graph of $f(x) = \dfrac{x^2 + 1}{x - 1}$, divide $x - 1$ into $x^2 + 1$:

Figure 3.38 The graph of $f(x) = \dfrac{x^2 + 1}{x - 1}$ with a slant asymptote

$$
\begin{array}{r}
\underline{1|}\quad 1 \quad 0 \quad 1 \\
1 \quad 1 \\
\hline
1 \quad 1 \quad 2
\end{array}
$$

Remainder

$$
x - 1 \overline{)x^2 + 0x + 1} \quad 1x + 1 + \frac{2}{x - 1}.
$$

Observe that

$$f(x) = \frac{x^2 + 1}{x - 1} = \underbrace{x + 1}_{} + \frac{2}{x - 1}.$$

The equation of the slant asymptote is $y = x + 1$.

As $|x| \to \infty$, the value of $\dfrac{2}{x - 1}$ is approximately 0. Thus, when $|x|$ is large, the function is very close to $y = x + 1 + 0$. This means that as $x \to \infty$ or as $x \to -\infty$, the graph of f gets closer and closer to the line whose equation is $y = x + 1$. The line $y = x + 1$ is a slant asymptote of the graph.

  In general, if $f(x) = \dfrac{p(x)}{q(x)}$, p and q have no common factors, and the degree of p is one greater than the degree of q, find the slant asymptote by dividing $q(x)$ into $p(x)$. The division will take the form

$$\frac{p(x)}{q(x)} = mx + b + \frac{\text{remainder}}{q(x)}.$$

Slant asymptote: $y = mx + b$

The equation of the slant asymptote is obtained by dropping the term with the remainder. Thus, the equation of the slant asymptote is $y = mx + b$.
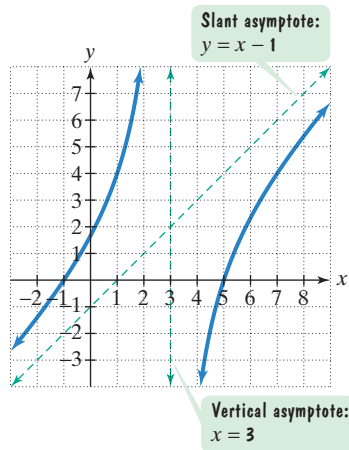
Slant asymptote:
$y = x - 1$

Figure 3.39 The graph of $f(x) = \dfrac{x^2 - 4x - 5}{x - 3}$

Vertical asymptote:
$x = 3$

EXAMPLE 8 **Finding the Slant Asymptote of a Rational Function**

Find the slant asymptote of $f(x) = \dfrac{x^2 - 4x - 5}{x - 3}$.

Solution Because the degree of the numerator, 2, is exactly one more than the degree of the denominator, 1, and $x - 3$ is not a factor of $x^2 - 4x - 5$, the graph of f has a slant asymptote. To find the equation of the slant asymptote, divide $x - 3$ into $x^2 - 4x - 5$:

$$\underline{3|}\ \ 1 \ \ -4 \ \ -5$$
$$\ \ \ \ \ \ \ \underline{3 \ \ -3}\quad \text{Remainder}$$
$$\ \ \ \ 1 \ \ -1 \ \ -8$$

$$1x - 1 - \dfrac{8}{x - 3}\quad\begin{array}{l}\text{Drop the remainder}\\\text{term and you'll have}\\\text{the equation of}\\\text{the slant asymptote.}\end{array}$$
$$x - 3\overline{)x^2 - 4x - 5}$$

The equation of the slant asymptote is $y = x - 1$. Using our strategy for graphing rational functions, the graph of $f(x) = \dfrac{x^2 - 4x - 5}{x - 3}$ is shown in **Figure 3.39**. ●

✓ **Check Point 8** Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$.

⑧ Solve applied problems involving rational functions.

Applications

There are numerous examples of asymptotic behavior in functions that model real-world phenomena. Let's consider an example from the business world. The **cost function**, C, for a business is the sum of its fixed and variable costs:

$$C(x) = (\text{fixed cost}) + cx.$$

Cost per unit times the number of units produced, x

The **average cost** per unit for a company to produce x units is the sum of its fixed and variable costs divided by the number of units produced. The **average cost function** is a rational function that is denoted by $\overline{C}$. Thus,

Cost of producing x units: fixed plus variable costs

$$\overline{C}(x) = \dfrac{(\text{fixed cost}) + cx}{x}.$$

Number of units produced

EXAMPLE 9 **Average Cost for a Business**

We return to the robotic exoskeleton described in the section opener. Suppose a company that manufactures this invention has a fixed monthly cost of $1,000,000 and that it costs $5000 to produce each robotic system.

a. Write the cost function, C, of producing x robotic systems.

b. Write the average cost function, $\overline{C}$, of producing x robotic systems.

c. Find and interpret $\overline{C}(1000)$, $\overline{C}(10,000)$, and $\overline{C}(100,000)$.

d. What is the horizontal asymptote for the graph of the average cost function, $\overline{C}$? Describe what this represents for the company.

Solution

a. The cost function, C, is the sum of the fixed cost and the variable costs.

$$C(x) = 1,000,000 + 5000x$$

Fixed cost is $1,000,000.   Variable cost: $5000 for each robotic system produced

b. The average cost function, $\overline{C}$, is the sum of fixed and variable costs divided by the number of robotic systems produced.

$$\overline{C}(x) = \frac{1,000,000 + 5000x}{x} \quad \text{or} \quad \overline{C}(x) = \frac{5000x + 1,000,000}{x}$$

c. We evaluate $\overline{C}$ at 1000, 10,000, and 100,000, interpreting the results.

$$\overline{C}(1000) = \frac{5000(1000) + 1,000,000}{1000} = 6000$$

The average cost per robotic system of producing 1000 systems per month is $6000.

$$\overline{C}(10,000) = \frac{5000(10,000) + 1,000,000}{10,000} = 5100$$

The average cost per robotic system of producing 10,000 systems per month is $5100.

$$\overline{C}(100,000) = \frac{5000(100,000) + 1,000,000}{100,000} = 5010$$

The average cost per robotic system of producing 100,000 systems per month is $5010. Notice that with higher production levels, the cost of producing each robotic exoskeleton decreases.

d. We developed the average cost function

$$\overline{C}(x) = \frac{5000x + 1,000,000}{x}$$

in which the degree of the numerator, 1, is equal to the degree of the denominator, 1. The leading coefficients of the numerator and denominator, 5000 and 1, are used to obtain the equation of the horizontal asymptote. The equation of the horizontal asymptote is

$$y = \frac{5000}{1} \quad \text{or} \quad y = 5000.$$

The horizontal asymptote is shown in **Figure 3.40**. This means that the more robotic systems produced each month, the closer the average cost per system for the company comes to $5000. The least possible cost per robotic exoskeleton is approaching $5000. Competitively low prices take place with high production levels, posing a major problem for small businesses.
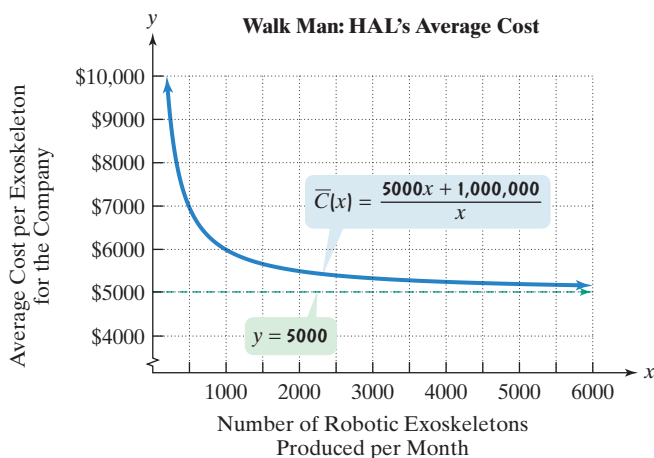

Walk Man: HAL's Average Cost

$\overline{C}(x) = \dfrac{5000x + 1,000,000}{x}$

$y = 5000$

Number of Robotic Exoskeletons Produced per Month

Average Cost per Exoskeleton for the Company

Figure 3.40

✐ Check Point **9** A company is planning to manufacture wheelchairs that are light, fast, and beautiful. The fixed monthly cost will be $500,000 and it will cost $400 to produce each radically innovative chair.

a. Write the cost function, C, of producing x wheelchairs.

b. Write the average cost function, $\overline{C}$, of producing x wheelchairs.

c. Find and interpret $\overline{C}(1000)$, $\overline{C}(10,000)$, and $\overline{C}(100,000)$.

d. What is the horizontal asymptote for the graph of the average cost function, $\overline{C}$? Describe what this represents for the company.
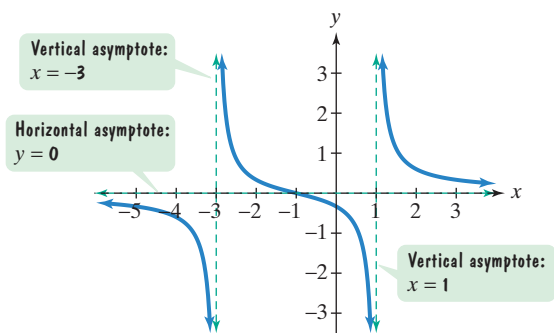
Exercise Set 3.5

Practice Exercises

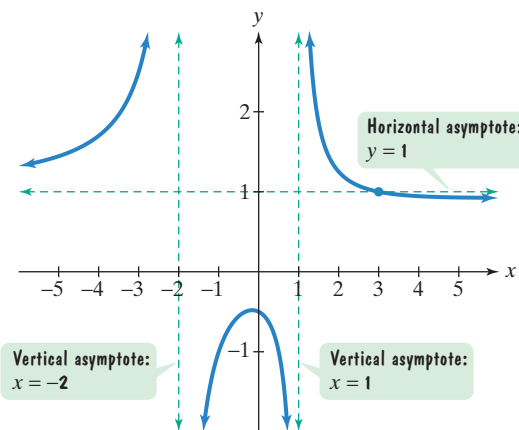*In Exercises 1–8, find the domain of each rational function.*

1. $f(x) = \dfrac{5x}{x-4}$

2. $f(x) = \dfrac{7x}{x-8}$

3. $g(x) = \dfrac{3x^2}{(x-5)(x+4)}$

4. $g(x) = \dfrac{2x^2}{(x-2)(x+6)}$

5. $h(x) = \dfrac{x+7}{x^2-49}$

6. $h(x) = \dfrac{x+8}{x^2-64}$

7. $f(x) = \dfrac{x+7}{x^2+49}$

8. $f(x) = \dfrac{x+8}{x^2+64}$

Use the graph of the rational function in the figure shown to complete each statement in Exercises 9–14.

9. As $x \to -3^-$, $f(x) \to$ _____.

10. As $x \to -3^+$, $f(x) \to$ _____.

11. As $x \to 1^-$, $f(x) \to$ _____.

12. As $x \to 1^+$, $f(x) \to$ _____.

13. As $x \to -\infty$, $f(x) \to$ _____.

14. As $x \to \infty$, $f(x) \to$ _____.

Use the graph of the rational function in the figure shown to complete each statement in Exercises 15–20.

15. As $x \to 1^+$, $f(x) \to$ _____.

16. As $x \to 1^-$, $f(x) \to$ _____.

17. As $x \to -2^+$, $f(x) \to$ _____.

18. As $x \to -2^-$, $f(x) \to$ _____.

19. As $x \to \infty$, $f(x) \to$ _____.

20. As $x \to -\infty$, $f(x) \to$ _____.

In Exercises 21–28, find the vertical asymptotes, if any, of the graph of each rational function.

21. $f(x) = \dfrac{x}{x+4}$

22. $f(x) = \dfrac{x}{x-3}$

23. $g(x) = \dfrac{x+3}{x(x+4)}$

24. $g(x) = \dfrac{x+3}{x(x-3)}$

25. $h(x) = \dfrac{x}{x(x+4)}$

26. $h(x) = \dfrac{x}{x(x-3)}$

27. $r(x) = \dfrac{x}{x^2+4}$

28. $r(x) = \dfrac{x}{x^2+3}$

In Exercises 29–36, find the horizontal asymptote, if any, of the graph of each rational function.

29. $f(x) = \dfrac{12x}{3x^2+1}$

30. $f(x) = \dfrac{15x}{3x^2+1}$

31. $g(x) = \dfrac{12x^2}{3x^2+1}$

32. $g(x) = \dfrac{15x^2}{3x^2+1}$

33. $h(x) = \dfrac{12x^3}{3x^2+1}$

34. $h(x) = \dfrac{15x^3}{3x^2+1}$

35. $f(x) = \dfrac{-2x+1}{3x+5}$

36. $f(x) = \dfrac{-3x+7}{5x-2}$

In Exercises 37–48, use transformations of $f(x) = \dfrac{1}{x}$ or $f(x) = \dfrac{1}{x^2}$ to graph each rational function.

37. $g(x) = \dfrac{1}{x-1}$

38. $g(x) = \dfrac{1}{x-2}$

39. $h(x) = \dfrac{1}{x} + 2$

40. $h(x) = \dfrac{1}{x} + 1$

41. $g(x) = \dfrac{1}{x+1} - 2$

42. $g(x) = \dfrac{1}{x+2} - 2$

43. $g(x) = \dfrac{1}{(x+2)^2}$

44. $g(x) = \dfrac{1}{(x+1)^2}$

45. $h(x) = \dfrac{1}{x^2} - 4$

46. $h(x) = \dfrac{1}{x^2} - 3$

47. $h(x) = \dfrac{1}{(x-3)^2} + 1$

48. $h(x) = \dfrac{1}{(x-3)^2} + 2$

In Exercises 49–70, follow the seven steps on page 373 to graph each rational function.

49. $f(x) = \dfrac{4x}{x-2}$

50. $f(x) = \dfrac{3x}{x-1}$

51. $f(x) = \dfrac{2x}{x^2-4}$

52. $f(x) = \dfrac{4x}{x^2-1}$

53. $f(x) = \dfrac{2x^2}{x^2-1}$

54. $f(x) = \dfrac{4x^2}{x^2-9}$

55. $f(x) = \dfrac{-x}{x+1}$

56. $f(x) = \dfrac{-3x}{x+2}$

57. $f(x) = -\dfrac{1}{x^2-4}$

58. $f(x) = -\dfrac{2}{x^2-1}$

59. $f(x) = \dfrac{2}{x^2+x-2}$

60. $f(x) = \dfrac{-2}{x^2-x-2}$

61. $f(x) = \dfrac{2x^2}{x^2+4}$

62. $f(x) = \dfrac{4x^2}{x^2+1}$

63. $f(x) = \dfrac{x+2}{x^2+x-6}$

64. $f(x) = \dfrac{x-4}{x^2-x-6}$